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Coupled Ising models with disorder

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Abstract. In this paper we study the phase diagram of two Ising planes coupled by a standard spin–spin interaction with bond randomness in each plane. The whole phase diagram is analysed with the help of Monte Carlo simulations and field theory arguments.

1. Introduction

In order to understand the role of impurities and inhomogeneities in real physical systems, many statistical models with quenched randomness have been proposed. The effect of randomness on continuous phase transitions has thus been of great interest for many years. The main question is to determine whether the randomness leaves the critical properties of the pure system unchanged. One prediction concerning models with random bonds arises from the Harris criterion [1], which states that bond randomness changes the values of the critical exponents only if the specific heat exponent α is positive. The effects of bond randomness have been studied intensively in the 2D Ising model and more recently in the $q < 4$ states Potts model. In the former case, the *log* singularity of the specific heat is transformed in a log(log) singularity [2], whereas in the latter case a new critical point with new critical exponents is reached [3]. Furthermore, it has been realized recently that a weak bond randomness can also have strong effects on 2D spin systems possessing a first-order transition. Following the earlier work of Imry and Wortis [4], Hui and Berker [5] have shown, using a phenomenological renormalization group, that bond randomness (even infinitesimal) induces a second-order transition (i.e. vanishing latent heat) in a system that would have undergone a first-order one. This result has been proved rigorously by Aizenman and Wehr [6]. At this level, a question naturally arises concerning the universality class of this second-order transition. In order to test these predictions, the q-states Potts model ($q > 4$ is necessary to obtain a first-order transition) with bond randomness has recently been extensively analysed numerically [7], the conclusion being that the transition is of second-order type due to disorder and the critical behaviour depends on q. Analytical results on the q-states Potts model are, as usual, much more difficult to obtain. However, this is not the case for those systems presenting what Cardy called a 'weak fluctuation-driven first-order transition' [8]. The typical example Cardy provided is a system of N critical Ising models coupled by their energy density. Without disorder, this perturbation drives the system in a strong coupling regime where perturbative renormalization group (RG) analysis fails. Yet, for this model, it can be shown non-perturbatively that this runaway flow is indeed associated with a massive regime (with a finite correlation length depending on the coupling). This is a way to define this 'weak fluctuation-driven first-order

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transition' phenomenon. The addition of weak bond randomness, even infinitesimal, makes the system flow to N decoupled critical Ising models [8]. This result has been extended to the case of N coupled Potts models [9, 10]. The result was a non-Ising-like second-order transition, the universality class depending on the sign of the coupling between the models. This class of examples is interesting because some infinitesimal bond randomness is able to change the 'weak fluctuation-driven first-order transition' into a continuous one. Concretely speaking, the RG runaway flow is completely turned over by disorder preventing a strong coupling regime. Nevertheless, a lot of questions still remain open. What happens if we take a different coupling from the energy density one? Does infinitesimal disorder will also change the order of the transition?

To bring some insight to these questions, we would like to study the following system: two Ising planes coupled by a spin–spin interaction where disorder (dilution or bond-randomness) is added in each plane. When the two Ising planes are critical, the spin–spin perturbation, which is much stronger that the energy–energy perturbation, drives the system in a strong coupling regime. It has been shown using integrable Toda field theories that a finite correlation length is generated [11] and the full spectrum of the theory can also be computed. In this sense, the situation is analogous to the model treated by Cardy [8], except that now the coupling is different and much stronger. Therefore, this model can bring some clues to the aforementioned questions. Moreover, it is worth noting that N Ising planes (with $N \gg 1$) coupled by the spin–spin interaction is a way to reach a 3D Ising model. $N = 2$ may be regarded as a kind of first step in this direction. Therefore, we expect the Aizenman–Wehr theorem not to apply here.

The plan of the paper is as follows. In section 2, we present analytical field theory arguments showing that a finite amount of disorder is necessary to change the 'weakly fluctuation-driven order first-order transition' into a continuous one. In section 3, we confirm it using Monte Carlo numerical simulations. In section 4, we summarize the results and discuss the whole phase diagram of the model.

2. Analytical results

2.1. Pure case

In this section, we first summarize results available in the pure case. The Hamiltonian then reads

$$H = -J \sum_{\langle i,j \rangle} \left(\sigma_i^1 \sigma_j^1 + \sigma_i^2 \sigma_j^2\right) - J' \sum_i \sigma_i^1 \sigma_i^2 \tag{1}$$

where $\sigma_i^{1,2} = \pm 1$, the first sum is over the nearest neighbours on a 2D square lattice, the same coupling J has been chosen for the two models and a coupling J' has been considered between the two planes. This model can be described in the continuum limit by the following action:

$$\mathcal{A} = \mathcal{A}_{\rm Is}^1 + \mathcal{A}_{\rm Is}^2 + m \int {\rm d}^2 x (\varepsilon_1(x) + \varepsilon_2(x)) + g \int {\rm d}^2 x\, \sigma_1(x) \sigma_2(x) \tag{2}$$

where $\mathcal{A}_{\rm Is}^i$ denotes the action of the critical Ising model, $\varepsilon_i(x)$ the Ising energy operator, σ_i the Ising spin operator, $m \propto (T - T_{\rm c})$ and $g \propto J'$.

When $g = 0$, the action corresponds to two decoupled massive Ising models which can be described by free massive Majorana fermions or by free Dirac fermions or by a sine-Gordon model. If the strongly relevant coupling g is switched on at the $m = 0$ critical point, the system is known to be driven into a massive regime whose mass spectrum has been computed exactly

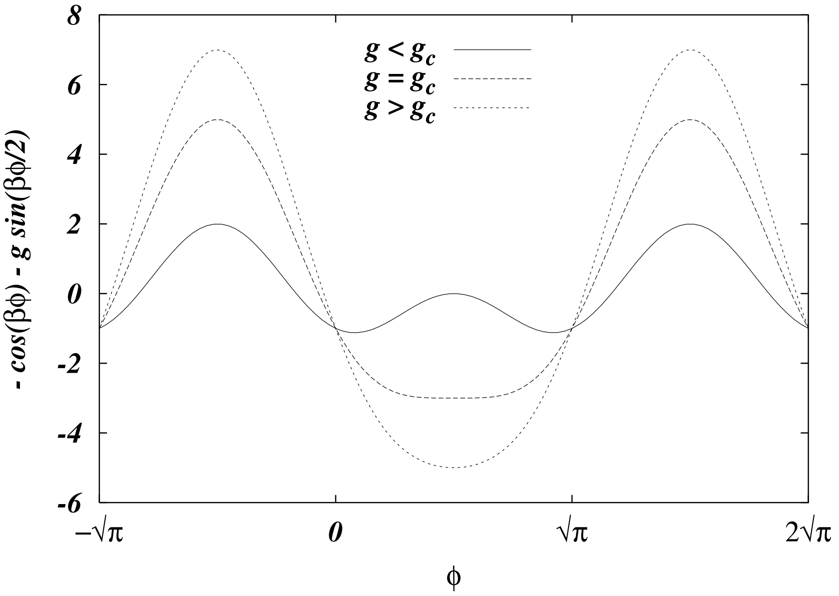

Figure 1. Evolution of the potential when varying g at fixed mass m. Criticality is reached for a critical value g_c.

[11]. When $g \neq 0$, the bosonic representation is more suitable (in terms of Majorana fermions representation, this interaction is indeed non-local), and the generalized sine-Gordon reads

$$\mathcal{A} = \int \mathrm{d}^2 x \, \tfrac{1}{2}(\nabla\phi)^2 + m\cos(\beta\phi) + g\cos(\alpha\phi \pm \delta) \tag{3}$$

with $\beta = \sqrt{4\pi}$, $\alpha = \sqrt{\pi}$ and $\delta = \frac{\pi}{2}$.

In order to obtain this expression, we write the Hamiltonian in terms of Majorana fermions and then use the standard bosonization formula (see, for example, [12] for a review). It is worth noting that the addition of a four-spin interaction such as $K\sum_{\langle i,j\rangle}\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2$ in the original Hamiltonian (1) (which therefore defines a generalized Ashkin–Teller model) would not change the form of the action (3) except the substitution $(\sqrt{4\pi}, \sqrt{\pi}) \longrightarrow (\beta, \frac{1}{2}\beta)$ with $\beta = \beta(K)$.

The action (2) is *a priori* difficult to study since it contains two strongly relevant perturbations and it is unfortunately non-integrable. Note that g is much stronger that m and scales around the critical point like $g \sim |m|^{7/4} \sim |T - T_c|^{7/4}$. A perturbed conformal field theory approach is inappropriate and beyond feasibility because the flow is quickly driven in a strong coupling regime.

The action (3) which is one particular case of the double-frequency sine-Gordon model has been qualitatively studied in the general case by Delfino and Mussardo [13]. It has been argued by looking at the solitonic structure in both weak and strong coupling regime ($m \gg g$ and $m \ll g$) that when the ratio $\frac{\alpha}{\beta} \equiv \frac{n'}{n}$ is a rational number and $\delta = \frac{\pi}{n}$, a phase transition should occur for $m > 0$ (i.e. $T > T_c$). Concerning our particular case, we can easily convince ourselves about this transition by looking at the evolution of the shape of the potential $\mathcal{V} = -m\cos(\beta\phi) - g\sin(\beta\phi/2)$ which passes from a periodic two-degenerate-minima to a periodic one-absolute-minimum situation when increasing the ratio g/m (see figure 1). From a Ginzburg–Landau point of view, it suggests that the transition is in the universality class of Φ^4, namely it is of Ising type.

Let us come back to the original Hamiltonian (1). In the limit $g \to \infty$, the spins σ_i^1 and σ_i^2 are locked to the same value and the model reduces to only one Ising model with a critical temperature $2J_c$. For a generic value of g the same scenario should occur for a critical temperature $J_c < T_c(g) < 2J_c$, since g tends to order the system in contrast to temperature. Therefore, there should be a critical line in the (T, g) plane interpolating between the fixed points $(J_c, 0)$ and $(2J_c, \infty)$. In the $g \gg J$ limit, we can rewrite the Hamiltonian (1) in the basis $\sigma_i = \sigma_i^1$; $\tau_i = \sigma_i^1\sigma_i^2$ and integrate over the τ variable using a mean-field approximation (we refer to [14] for a more detailed analysis in a different context). It remains *one* Ising model with approximate effective coupling $2J(1 - cJ^2/g^2) \approx 2J_{\text{eff}}$, where c is a positive number satisfying a self-consistent equation. The Ising criticality is reached when $J_{\text{eff}} = J_c$. This analysis is valid only close to the $(2J_c, \infty)$ Ising fixed point, namely in the strong-coupling limit.

According to Zamolodchikov's c-theorem [15] (without disorder the theory is still unitary), the flow goes from the $(J_c, 0)$ fixed point with central charge $c = 1$ toward the ending Ising fixed point $(2J_c, \infty)$ with central charge $c = \frac{1}{2}$. From the continuum limit point of view, it is worth noting that the elementary excitations around this fixed point have nothing to do with those around the $c = 1$ fixed point (the correspondence is indeed non-local, explaining why a standard perturbation theory around the $c = 1$ fixed point is hopeless).

2.2. Disordered case

Let us now add bond randomness or dilution in each Ising plane. In the two limits $g = 0$ and $g = \infty$ the problem reduces to the well known disordered Ising model and there are no new fixed points. Can we say something in the general situation? The phase diagram is parametrized by three variables, T, g and Δ the strength of the disorder. In the planes $\Delta = 0$ and $g = 0$, the shape of the transition separating the ferromagnetic phase from the paramagnetic one is known, respectively, from results concerning the disordered Ising model and from the pure case analysis described above. Therefore, by a continuity argument, we expect a critical surface joining the two Ising critical lines. Let us focus on the $T = T_c$ plane. In the continuum limit, the effect of disorder is to change m into $m(x)$ in the action (3) with $\overline{m(x)} = 0$, $\overline{m(x)m(x')} = \Delta\delta(x - x')$ for a Gaussian disorder. This action is very difficult to study with the usual tools and we can only give qualitative arguments. At a fixed value of g, when $m(x) > m^*(g) = g/4$, it implies a local high-temperature region surrounded by low-temperature regions. In the limit $\Delta \gg m^*(g)$ (still at fixed g), the phenomenology of the critical disordered Ising model is recovered with a mixture of low- and high-temperature regions. This suggests that there should exist a finite critical value $\Delta^*(g)$ of the disorder strength able to disorder the system. In fact, the situation resembles that of a disordered Ising model with $\overline{m(x)} = m_0 > 0$. This is clearly a non-perturbative phenomenon. We conjecture this transition to be of Ising type governed by the Ising fixed point (with $c = \frac{1}{2}$) at $(2J_c, \infty, \Delta = 0)$ with a log(log) behaviour for the specific heat due to the disorder. Another non-trivial fixed point at $\Delta^* > 0$ cannot *a priori* be excluded but is very improbable according to the symmetries of the model under consideration. Note that, in contrast to the examples treated in [8–10], where some infinitesimal disorder changes the weakly driven first-order transition into a continuous one, here a *finite* value of the disorder is needed!

3. Numerical analysis

To put these qualitative arguments on firmer grounds, we have performed Monte Carlo simulations at $T = T_c$. Let us present our numerical results. We have considered two 2D

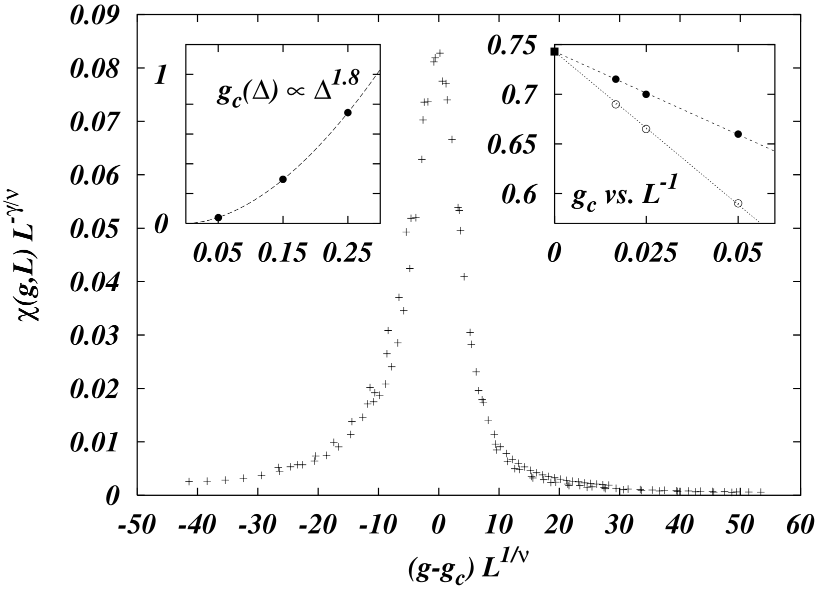

Figure 2. Susceptibility data rescaled with pure 2D Ising model exponents: $\nu = 1$ and $\gamma = \frac{7}{4}$. In the left-hand inset we report the values for the critical coupling g_c versus the disorder amount Δ, together with the best power-law fit. In the right-hand inset we show, as a function of $1/L$, the g values where the specific heat (full symbols) and the susceptibility (open symbols) have a peak. Here $\Delta = 0.25$ and the Binder cumulants crossing point is $g_c = 0.743$ (full square).

Ising models, with bond dilution Δ in both planes, i.e. $P(J) = \Delta\delta(J) + (1 - \Delta)\delta(J - 1)$. The disorder is independent in each plane. We let the planes interact via a spin–spin coupling term, the strength of which is g.

We have performed numerical simulations with three different lattice sizes ($L = 20, 40, 60$) and for three values of the disorder ($\Delta = 0.05, 0.15, 0.25$). Lattice sizes may seem small compared with previous studies on the disordered Ising model [16], however, they seem to be in the scaling regime (see below) and so the extrapolation to large sizes is safe, at least up to the precision we are interested in. For any disorder value we have found clear evidence that, on increasing the value of the coupling g, the system undergoes a phase transition from a paramagnetic to a ferromagnetic phase. The critical coupling values can be well fitted by a power law: $g_c(\Delta) \propto \Delta^{1.8}$ (see the left-hand inset in figure 2, where the errors are smaller than the symbol size). Note that the value of this exponent is very close to the theoretical one obtained by scaling arguments close to the $c = 1$ fixed point, which is $\frac{7}{4}$.

For every Δ, the critical coupling g_c has been determined as the crossing point of the Binder cumulants. This crossing point has almost no finite-size effects. Two more estimations of g_c can be obtained from the g values where the specific heat and the susceptibility have a peak. In fact, these values converge to g_c in the large-L limit. However, they show much larger finite-size effects than the Binder cumulants crossing point. Then, in order to obtain a better estimation of g_c, we have used the data for every lattice size and we have considered finite-size corrections through the formula $g_c(L) = g_c - AL^{-1/\nu}$, with $\nu = 1$. We have found that our data are compatible with that formula for any disorder value and the g_c estimation coincides with that extracted from the Binder cumulants (see the right-hand inset in figure 2). The effect of increasing Δ simply reflects in a larger value for the non-universal coefficient A.

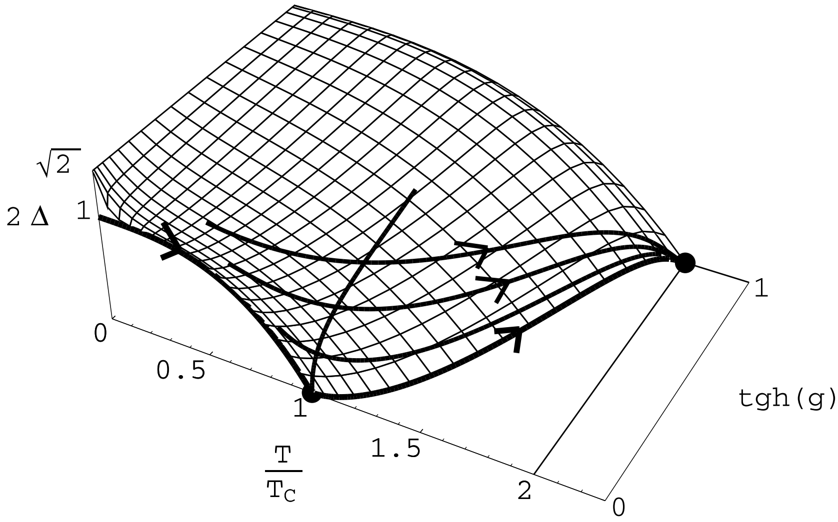

Figure 3. Phase diagram in the variables T, g, Δ, where Δ is the bond dilution. The critical surface separates the ferromagnetic region (below) from the paramagnetic one (above). The arrows indicate the flow direction. We have numerically studied the system along the line $T = T_c$ shown in the figure.

In order to verify that the transition is governed by the pure 2D Ising model fixed point, we have analysed our numerical data using finite-size scaling and the critical exponents of the pure 2D Ising model, i.e. $\nu = 1$ and $\gamma = \frac{7}{4}$. We have found very good data collapse for any disorder value Δ. In particular, in figure 2 we show the collapse of the rescaled susceptibility for the larger disorder value $\Delta = 0.25$. For smaller disorder amplitudes the scaling is even better and the scaling function is more peaked, signalling that the critical region is narrower.

4. Discussion and conclusions

In this paper we have presented analytical and numerical evidence that in two coupled 2D Ising models the 'weak fluctuation-driven first-order transition' changes to a continuous one only when a finite amount of disorder is added. Moreover, we have seen that, even for finite and large disorder amplitudes, the phase transition is in the 2D disordered Ising universality class (Ising transition modified by logarithm corrections). The results we obtained are summarized in the schematic 3D phase diagram depicted in figure 3. Our analysis strongly suggests that almost all the critical surface shown in figure 3 is governed by the 2D Ising fixed point, except the curve in the $g = 0$ plane (governed by the $c = 1$ fixed point) and that in the $T = 0$ plane (governed by a percolation fixed point†). Because of the known flow lines on this surface (those with an arrow in figure 3), we believe that any transition across the surface is governed by the $c = \frac{1}{2}$ fixed point located in $(2J_c, \infty)$ (the large dot on the right-hand side of figure 3) because the $c = 1$ fixed point is strongly repulsive in the g direction (even if it is attractive on the $g = 0$ plane [17]). For an even clearer numerical check, a numerical estimation of the central charge would be welcome.

† Note that the $T = 0$ critical line is discontinuous at $g = 0$, as can been seen in figure 3. In fact, for every $g \neq 0$ the bond percolation threshold is $1/\sqrt{2}$, while for $g = 0$ is $\frac{1}{2}$

As a conclusion, we would like to mention that several directions remain open. First of all, a theoretical framework to describe quantitatively such non-perturbative transition is needed. Secondly, it would be interesting to extend this analysis for $N > 2$ coupled Ising models, and finally it is worth performing a similar analysis for correlated disorder.

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